Math 128
Lagrange Examples – September 12, 2008

1. **Example:** Let $z = xy$. Find all maximums and minimums of z on the circle $x^2 + y^2 = 1$.

We apply the technique of Lagrange multipliers.

First, we rewrite the constraint $x^2 + y^2 = 1$ in the form $g(x, y) = 0$, that is:

$$x^2 + y^2 - 1 = 0.$$

Second, we take the Lagrange multiplier function $F(x, y, \lambda) = f(x, y) + \lambda g(x, y)$. In this case:

$$F(x, y, \lambda) = xy + \lambda(x^2 + y^2 - 1).$$

Finally, we find the critical points of this function F of three variables. Lagrange's Theorem says that the x and y coordinates will be the critical points of f. There are three derivatives:

$$\begin{aligned} F_x &= y + 2\lambda x \\ F_y &= x + 2\lambda y \\ F_\lambda &= x^2 + y^2 - 1. \end{aligned}$$

(Notice in passing that the third derivative is exactly the constraint that we started with!)

Finally, we set $F_x = F_y = 0$, and work some algebra. Since $F_x = 0$, we have

$$2\lambda x = -y,$$

so unless $x = 0$, we have $-\lambda = \frac{y}{2x}$. Similarly, the second equation gives us $-\lambda = \frac{x}{2y}$. Setting these equal, we get

$$\frac{y}{2x} = \frac{x}{2y}.$$

We multiply through by $2xy$, which gives us

$$2xy \cdot \frac{y}{2x} = y^2 = 2xy \cdot \frac{x}{2y} = x^2,$$

and in short, that

$$x^2 = y^2, \text{ i.e. } x = \pm y.$$

but

$$x^2 + y^2 = x^2 + x^2 = 2x^2 = 1,$$

and so $x = \pm\sqrt{2}$.

We recall that we did not eliminate the possibility that $x = 0$ or $y = 0$ above, when we divided. The possible critical points, and values of f are summarized in the following table:

x	y	$f(x,y) = xy$
0	± 1	0
± 1	0	0
$\sqrt{2}$	$\sqrt{2}$	2
$\sqrt{2}$	$-\sqrt{2}$	-2
$-\sqrt{2}$	$\sqrt{2}$	-2
$-\sqrt{2}$	$-\sqrt{2}$	2

We conclude that $f(x,y) = xy$ has a maximum value of 2 on the circle $x^2 + y^2 = 4$, obtained at the points $(\sqrt{2}, \sqrt{2})$ and $(-\sqrt{2}, -\sqrt{2})$; and a minimum value of -2, obtained at the points $(-\sqrt{2}, \sqrt{2})$ and $(\sqrt{2}, -\sqrt{2})$.

2 **Example:** An open-top cardboard box has volume 4000. What are the height, width, and depth that minimize its surface area?

We set up the problem. Let h, w, and d be height, width, and depth, respectively. Then

$$\text{Volume} = 4000 = hwd$$

and

$$SA(h,w,d) = wd + 2hw + 2hd.$$

We apply the technique of Lagrange multipliers.

First: we rewrite the constraint in the form $hwd - 4000 = 0$.

Second: we write down the function

$$F(h,w,d,\lambda) = wd + 2hw + 2hd + \lambda(hwd - 4000).$$

Third, we examine critical points of this function. Start by taking the derivatives

$$F_h = 2w + 2d + \lambda wd$$

$$\begin{aligned} F_w &= d + 2h + \lambda hd \\ F_d &= w + 2h + \lambda hw \\ F_\lambda &= hwd - 4000. \end{aligned}$$

We set $F_h = F_w = F_d = 0$, and solve in each for λ. In F_h, we have

$$\begin{aligned} 2w + 2d + \lambda wd &= 0 \\ \lambda wd &= -2w - 2d \\ \lambda &= \frac{-2w - 2d}{wd} \\ \lambda &= -\frac{2}{d} - \frac{2}{w}. \end{aligned}$$

In F_w and F_d we go through similar steps, and find that

$$\begin{aligned} \lambda &= \frac{-d - 2h}{hd} = -\frac{1}{h} - \frac{2}{d} \\ \lambda &= \frac{-w - 2h}{hw} = -\frac{1}{h} - \frac{2}{w}. \end{aligned}$$

We have three things that equal λ now, and we set them equal to each other and solve. First: $-\frac{1}{h} - \frac{2}{d} = -\frac{1}{h} - \frac{2}{w}$. We cancel the $\frac{1}{h}$'s, and are left with $-\frac{2}{d} = -\frac{2}{w}$. Cancelling the -2's and taking the reciprocal of both sides gives us that $d = w$.

A similar procedure is followed with $-\frac{1}{h} - \frac{2}{d} = -\frac{2}{d} - \frac{2}{w}$. We cancel the $\frac{2}{d}$'s, giving us $-\frac{1}{h} = -\frac{2}{w}$, and hence $w = 2h$.

Since $w = d = 2h$, and we get that $hwd = h \cdot 2h \cdot 2h = 4h^3 = 4000$, thus, that $(h, w, d) = (10, 20, 20)$ is a critical point. This critical points is, in fact, a minimum.